

Performance control for efficient design of double-layer grids under uniform loading

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Abstract Performance control (PC) is a recently developed, observation based, design process that aims at rational and efficient selection of structural elements rather than investigating their usefulness through iterative processes. The basic notion behind PC is that structural response is mainly a function of design and detailing, rather than numerical computations. The design concept uses pre-selected target displacements and failure mechanisms as key control objectives. While elastic analysis is still the predominant design methodology for double-layer grids (DLG) and similar structures under service conditions, performance-based procedures could further enhance their functional uses up to plastic limit states. The current contribution presents a general technique for the estimation of maximum plastic displacements of twistless, orthotropic, DLG of regular formation, simply supported along the sides of a parallelogram, and subjected to monotonically increasing uniform distribution of normal nodal forces. In the interim, a new application for the uniqueness theorem has also been discussed. The proposed methodology is ideally suited for preliminary design as well as capacity analysis purposes. It lends itself to both manual and spreadsheet computations. The applications of the proposed solutions have been illustrated through a number of generic examples. The paper does not address seismic conditions.

Keywords Performance control · Double-layer space trusses · Plastic design · Load sharing concept · Displacements at incipient failure · Uniqueness theorem

Introduction

The plastic limit state behavior of flexural grillages and double-layer articulated grids has been the subject of many in-depth analytical and experimental studies since the early 1950s. The pioneering efforts in these fields are due to, among others Hayman (1952, 1953), Stevens (1961, 1968), Hongladaromp et al. (1968), Wah (1969), Grigorian (1971, 1972, 1973a) and Saka and Heki (1971). More definitive treatments have also been reported, among others, by Grigorian (1973b), Marsh (1975, 1977), Supple and Collins (1981), Parke and Walker (1984), Schmidth (2000a), Kaveh and Talatahari (2009), Ghoizadeh et al. (2012) and Maalek and Abadi (2012). A well-organized review of space structures with particular emphasis on analytic methodologies including plastic limit analysis has been compiled by Heristchian (2000).

While the space structures industry has witnessed remarkable advances in both the technological and computational aspects of such systems, the same cannot be claimed for the corresponding design methodologies as a whole. An excellent account of space structural innovations may be found in a comprehensive bibliography by Schmidth (2000b).

The current presentation does not aim at discussing the merits of space structures, but rather to introduce the basis of a new design philosophy that might stimulate and advance the customary thinking on the subject. The results of the afore-mentioned studies point to two important design considerations that; the successful plastic limit state design (LSD) of such structures can be achieved provided that the ductility of the tension members can be preserved and that the buckling of compression members can be delayed, through proper sizing and detailing, until after the yielding of preselected groups of tension elements, and that

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the out of plane displacements of the structure at the onset of collapse can be accurately estimated.

Despite their technical limitations, allowable stress (ASD), LSD, load resistance factor design (LRFD) and, to a lesser extent, plastic design (PD) are still the most widespread methods of space structure design world wide, e.g., Davison and Owens (1993), Cuoco (1997) and Ramaswamy et al. (2002). While these methodologies have served the engineering communities well, at the expense of the consumer, they result in almost identical final products. The main weakness of the ASD is in its limitation to the elastic range of response and at best to the onset of plasticity. LSD and LRFD are essentially rationalized elastic methodologies and may at best be regarded as lower bound plastic designs. PD tends to produce more economic results (Beedle 1958; Neal 1963; Nethercot 2001), but is also limited in nature, depends on the correct prediction of the failure mechanism and does not address service level behavior (Heyman 1961). The fallacy that PD cannot be used to estimate displacements at incipient collapse, together with the dominating influences of electronic computations has hindered the progress of the more rational performance-based plastic designs, such as the now well-recognized push-over analyses (Naeim 2001; Bozorgnia and Bertero 2004) and similar methods of approach. But in fairness, the availability of the same electronic facilities has helped to develop a large number of highly sophisticated analytical and optimization methodologies for almost all types of engineering structures, e.g., Nooshin (1981, 1985) and Kaveh et al. (2011). However, the recently introduced purpose-specific PC (Grigorian and Grigorian 2012a, b, c), which is a more comprehensive design procedure, embraces the merits of both the elastic and plastic methods of approach. PC utilizes the complete elastoplastic













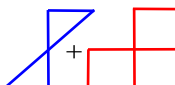
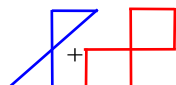
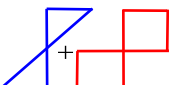
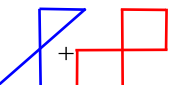
characteristics of ductile structures, including plastic deformations at incipient collapse, as part of the inclusive design strategy.

A symbolic and self explanatory comparison of the prevailing methods of design; ASD, LRFD and PD and the proposed PC is presented in Table 1, where it may be observed that the most significant difference between PC and the traditional methods of design is that in PC the design of the structure is based on the performance of the entire structure rather than the capacity of the weakest member of the system. The purpose of the current article besides presenting generalized failure load formulae for simply supported DLG is to propose PC as a rational alternative for space structure design under gravity loading conditions.

Performance control

While performance-based plastic design procedures have long been identified as the preferred methods of seismic design and other extreme loading conditions (Housner 1960; Mazzolani and Pilosu 1996; Goel et al. 2010; Grigorian 2013a, b), they have not been recognized fully as viable means of design for common types of space frames and similar structures under non-seismic scenarios. The main difference between performance-based plastic design for earthquake resistant structures and PC is that the former depends on plastic rotations beyond the formation of the last plastic hinge, i.e., the perceived ductility limit, whereas the latter relies on the more definitive displacements at incipient collapse. On this basis, the author contends that; barring instabilities, the behavior of any ductile structure can be defined by at least two observable, characteristic

Table 1 Symbolic comparison of current design philosophies

| Design | Basic analysis | Force level | Displacements | Selection | Comments | Defining stage | Design basis |
|---------|---|---|---|--|---------------------------------|------------------------------------|-----------------|
| ASD |  |  |  |  | Purely elastic lower bound | First yield | Weakest member |
| LRFDLSD |  |  |  |  | Essentially elastic lower bound | Undefined | Weakest member |
| PD |  |  |  |  | Purely plastic lower bound | Incipient collapse | System strength |
| PC |  |  |  |  | Elasto-plastic unique | First yield and incipient collapse | System behavior |



limits, one at first yield and the other at incipient collapse. The knowledge associated with the failure of the first and last plastic hinges in beams or yielding members in trusses can furnish valuable information for the rational design of all such frameworks.

PC is a design philosophy in which the properties of groups of members are selected in accordance with pre-determined performance-based objectives, such as limiting displacements at first yield and/or at incipient collapse, rather than being investigated against consensus-based criteria. PC is also a precursor to developing structures of uniform response (Grigorian 2011), where the individual elements of the same group of members can also be curtailed to simultaneously undergo equal stresses and strains throughout the structure. Structures of uniform response act as frameworks of equal strength and stiffness in which members of the same group share the same demand–capacity ratios regardless of the number of elements within the system.

In PC, failure patterns and stability conditions are enforced rather than tested. Most importantly, it enables the engineer to control the design rather than check design-related numerical output. PC appears to be the most desirable design method for such space frames as steel and reinforced concrete grillages, interconnected vierendeel girders, DLG and moment frames. PC procedures result in highly predictable structural behavior and economically efficient designs for the class of regular, space frames considered in this paper.

Neither elastic nor plastic analysis alone can provide insight into the complete elastic–plastic behavior of space frames under monotonously increasing forces. However, PC being a comprehensive method of approach embraces the beneficial and well-tested principles of all three traditional methods of approach into a single, rational method of design for all types of structures under normal as well as extreme loading conditions. In short, PC aims at satisfying the lower bound/elastic conditions at service loading and the uniqueness requirements at extreme loading conditions. In fact it has been shown (Grigorian 2013c) that the entire loading history of the subject systems, i.e., the idealized elastic–plastic force–displacement relationship starting from zero up to first yield, from first yield up to incipient collapse and the upper ductility limit can be constructed using only two points of reference.

The following example has been devised to demonstrate the applications of PC to the design of a typical grillage problem, where it has been shown that in PC, control of displacements at two essential stages and the collapse mechanism is built into the design process from the very start, rather than investigating relative magnitudes of iterated numbers to arrive at a final solution.

Example 1-a typical application of performance control

As an introductory example consider the controlled design of the simply supported, 4×4 isotropic, regular, twistless, flexural grillage of Fig. 1a under uniform distribution of normal nodal forces P , such that its maximum central displacements do not exceed $L/360$ at first yield and $L/240$ at incipient collapse. The corresponding theoretical elastic–plastic load–displacement curve is presented in Fig. 1b.

Solution Assume the load factor at collapse = 1. Since $m = n = 4$ and $L = 4a$, then the complete long hand elasto-plastic analysis for this particular case (Grigorian 2013b) gives $P_Y = M^P/1.1716a$, $P_C = M^P/a$, $\delta_Y = 1.8974P_Y a^3/EI$ and $\delta_C = 2.5P_C a^3/EI$. Here P_Y and P_C , and δ_Y and δ_C represent the load and displacement limits at first yield and incipient collapse, respectively, M^P and E stand for plastic moment of resistance and young's modulus, respectively. Therefore, the controlling section inertia, I , would correspond to either $I \geq 1.8974P_Y a^3/\delta_Y E = 360 \times 1.8974 M^P a^3/1.1716a \times 4a = 145.75aM^P/E$, or $I \geq 2.5P_C a^3/\delta_C E = 240 \times 2.5 M^P a^3/a \times 4a = 150aM^P/E$. Obviously $M^P = Pa$ and $I = 150aM^P/E$ govern the controlled design of the subject grillage. Note that the proposed solution is in conformance with both the elastic and plastic design requirements, and that no further checking is necessary.

The two distinct displacement values Y_Y at P_Y and Y_C at P_C provide sufficient information for the construction of an accurate load–displacement curve OYC as depicted in Fig. 1b for the grid under consideration. The tri-linear curve $OYCP$ also includes the symbolic ductility limit Y_P at P . It may be observed that for the particular example $Y_C \approx 1.54Y_Y$ and that all three points Y , C and P represent definitive design limits as well as reliable PC criteria. However, the approximate, bilinear curve $OY'CP$ can also be used for equally reliable design and/or investigative purposes, without resorting to complicated analysis for intermediate values of $Y_Y < Y < Y_P$, corresponding to $P_Y < P < P_C$.

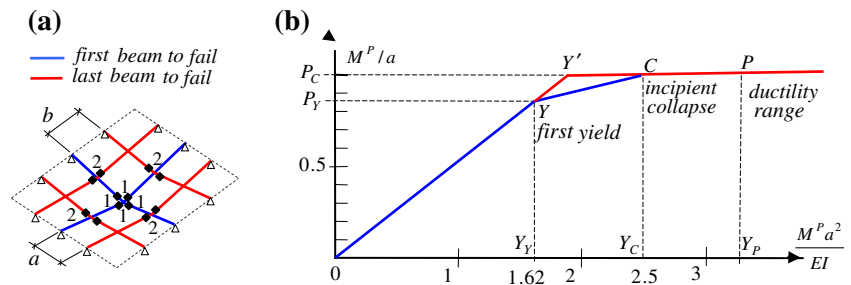
While segments OY' , $Y'C$ and CP of the load–displacement curves may be associated with ASD, LRFD and PD philosophies, respectively, the same segments may also be used to estimate percentage damage, assess global integrity or to propose intermediate control criteria, such as restricting maximum displacements to $(L/400)$ at 50 % first yield or to $(L/200)$ at 75 % of the collapse load, etc. Any design based on Y_C and P_C might still be conservative due to safeguards involved in PC and that Y_P associated with the ductility limit could be much larger than Y_C .

Performance control for double-layer space truss systems

An understanding of the ultimate load behavior of DLG in addition to their elastic response is a priori to establishing



Fig. 1 **a** 4×4 Grid and failure mechanism, **b** elastic–plastic load–displacement curve



control criteria for all loading phases of the structure. The key to the successful implementation of PC in multimember space structures is in regulating the responses of groups of similar elements such as the joints, upper and lower chords and diagonal members.

The purpose of this section, therefore, is to establish sets of closed form equations for the internal forces of the system that are in conformity with the requirements of the uniqueness theorem (Heyman 1971). This is achieved by first defining the basic design objectives, the underlying assumptions and the controlling design criteria, and then deriving the constituent equations of equilibrium in terms of the finite difference operators.

The general scheme of a regular, rectangular pyramid base, simply supported, DLG under a uniform distribution of normal nodal forces and the preferred failure mechanism involving the plastic extensions of the lower chord members is shown in Fig. 2. In this scheme, m and n , and, a and b represent number and length of bays in X and Y directions, respectively.

The mechanics of the less desirable collapse modes involving the failure of the upper chord and/or diagonal members has been investigated, among others by Grigorian and Lashkari (1973) and Schmidh (2000a) and is not included as part of the current study.

An enlargement of a representative segment of the structure together with the corresponding disposition of the

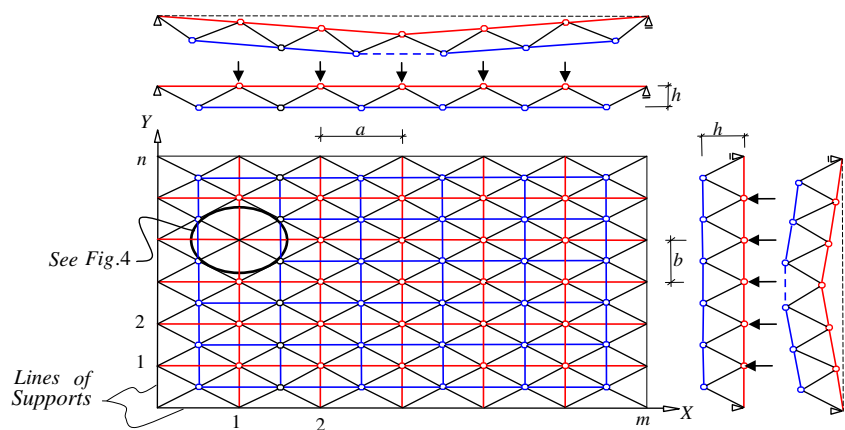
member forces acting on two neighboring joints is presented in Fig. 4.

Basic design objectives

The mathematical formulation of PC is based on the implementation of the following design objectives;

1. The maximum out of plane displacements of the structure can be controlled during all phases of the loading, starting from zero to first yield, and from first yield to incipient collapse.
2. Buckling failure is prevented throughout the structure. Global collapse (mechanism) through yielding of the lower chord members is reached if the concept of no compression failure is observed.
3. The proposed design should satisfy all conditions of the uniqueness theorem.
4. For minimum weight condition, the demand–capacity ratios of all members can be selected as close to unity as possible, in which case the displacement analysis should be reiterated for reduced cross sections.
5. The design load factors shall be in conformance with the pertinent code requirements.
6. The structure shall not fail prematurely due to the formation of inactive plastic deformations.

Fig. 2 Regular, double-layer space truss, layout and preferred failure mechanism under uniform normal nodal loading



7. The prescribed target displacements at working stress and at incipient collapse are not exceeded.
8. All plastic displacements can take place without any restricting effects from the connections or cladding components.
9. The differences between nodal and corresponding inter-nodal displacements can be ignored for practical design purposes.
10. Connection and support failure are prevented under all loading conditions.

Basic design assumptions

The methodology expounded in this presentation is based on the following design assumptions:

1. There are no initial imperfections, lack of fit or settlement of supports.
2. Axial deformations are small and do not affect the geometry of the structure.
3. All members resist purely axial forces.
4. The uniform normal nodal forces increase monotonously from zero to ultimate.
5. Initial design is based on identical section properties for each group of elements.
6. The joints are made out of inextensible materials.
7. The possible benefits of strain hardening and yield over-strength can be ignored.
8. The self weight of the structure can be included as part of the normal nodal force distribution.
9. Ductile failure can only take place within the lower chord (tension) members.
10. The frames are three-dimensional structures and are constructed out of ductile materials.

Basic design criteria

The subject structure is composed of three types of nominal prismatic members: the upper chords, the lower chords and the diagonals. These members can be under either direct tension or compression. The prevailing yield criteria may be simplified as:

$$T_{x,y} \leq T, \bar{T}_{x,y} \leq (\mu_t T = \bar{T}), \bar{C}_{x,y} \leq (\mu_c C = \bar{C}), \\ (F_{x,y}, R_{x,y}, Q_{x,y} \text{ and } W_{x,y}) \leq S_t \text{ and/or } S_c \quad (1)$$

where, $C_{x,y}$ and $\bar{C}_{x,y}$, and, $T_{x,y}$ and $\bar{T}_{x,y}$ represent the upper and lower chord forces in the X and Y directions, respectively. $F_{x,y}$, $R_{x,y}$, $Q_{x,y}$ and $W_{x,y}$ are the internal forces of the diagonals meeting at a common joint xy . Similarly, C and \bar{C} , and, T and \bar{T} stand for the ultimate load capacities of the

upper and lower chord members in the X and Y directions, respectively. S_t and S_c are the ultimate tensile and compressive load capacities of the diagonal members, respectively. C , \bar{C} and S_c can best be compared against either the Melbourne regression formula (Stevens 1961, 1968) or (Merchant's 1958) semi-empirical relationship presented in Appendix 1. μ_t and μ_c are the coefficients of orthotropy for the horizontal tensile and compressive members, respectively.

Plastic limit state analysis

PC is based on the full understanding of the complete elasto-plastic behavior of the structure. The true plastic collapse of any ductile system can only be assessed through the satisfaction of the requirements of the uniqueness theorem. Unfortunately, the uniqueness theorem does not directly address elasto-plastic deformations at incipient collapse.

However the author contends that uniqueness theorem can also be used to determine exact maximum displacements at collapse (Grigorian 2013b, c).

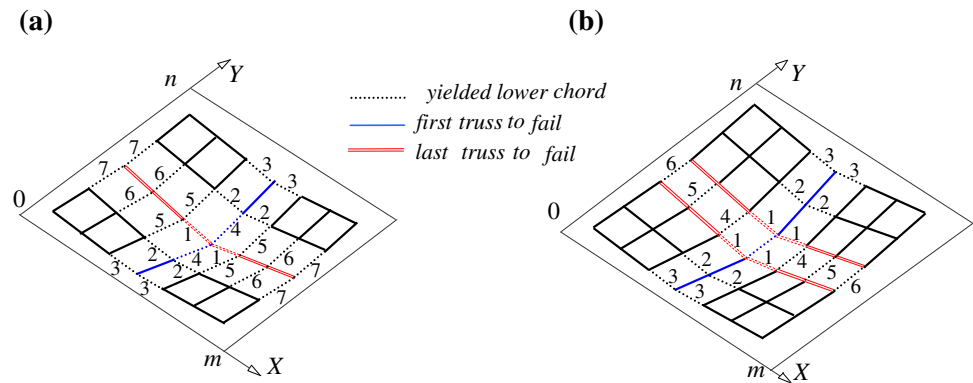
In the forthcoming sections, the uniqueness theorem is first utilized to establish the true distribution of the internal forces at collapse. An attempt is then made to relate the maximum plastic out of plane displacements to the corresponding distribution of internal stresses. A brief description of the uniqueness theorem is provided as part of the current discussions.

Plastic failure load analysis

The plastic failure load of the subject class of structures can be deducted either by long hand analysis or directly using the principles of the load sharing concept (LSC) (Grigorian and Yaghmai 1973). The LSC is briefly reintroduced as both a method of direct analysis and a means of verifying the results of long hand solutions. The long hand closed form solutions are then utilized to compute the corresponding displacements at incipient collapse. It is quite common, in multimember space trusses, to encounter conditions of over-collapse involving several extensionally inactive yielded elements that neither participate nor contribute toward the development of the carrying capacity of the system. Nevertheless, for the purposes of ultimate load studies, it is sufficient to engage only the active plastic hinges that are needed to generate kinematically admissible collapse patterns, without due regards to the sequence of formation of active and/or inactive sets of yielding members. However, since the numbers of bays, m and n ,



Fig. 3 **a** Lower chord failure pattern, m odd n odd. **b** Lower chord failure pattern, m and n even



whether even or odd, influence the shape of the failure patterns and affect the magnitude of the corresponding failure loads for sparsely meshed space frames, it is necessary to study all combinations of such collapse mechanisms. Figure 3a, b depicts two such plausible failure patterns associated with the plastic stretching of the lower chord elements. In the forthcoming study, only the analysis of one such space truss with even number of bays in both directions is presented in some detail. For the sake of brevity, the analysis of other cases being repetitious is not presented as part of this study. A graphical representation of a typical first failing truss with both even and odd number of bays is presented in Fig. 5.

The load sharing concept

The LSC was originally developed to study the plastic failure of regular rectangular twistless gridworks of uniform strength under static, continuous, normal nodal loading. Gridworks of uniform ultimate strength are a class of three-dimensional structures in which the variation of the plastic moment of resistance of the beams of any of the two sets is the same as the variation of the representative intensity of the nodal loading acting on the same beams (Grigorian et al. 1975). It has been shown that LSC can be further utilized as a generalized method of displacement analysis at incipient collapse. The LSC for the particular class of grillage types discussed herein, with all combinations of basic boundary support conditions, states that:

The collapse load intensity, P_u of the twistless grillage system, composed of flexural and/or articulated elements, is given by the sum of the collapse load intensities P_x and P_y of any two such intersecting beams or trusses, i.e.

$$P_u = P_x + P_y \quad (2)$$

Designs based on the LSC meet the requirements of the uniqueness theorem and as such cannot be far from minimum weight solutions (Faulks 1954a, b; Grigorian 2013a).

The load sharing concept and maximum displacements at incipient collapse

The LSC as described by Eq. (2) suggests that if the failure load of the grillage of twistless systems can be computed as the sum of the failure loads of two such intersecting beams, trusses, Vierendeel girders, etc., then the magnitude of the corresponding displacements should also be related to the sum of the same two structural systems. In other words, it implies that;

The maximum transverse displacement of the subject structure at incipient collapse may also be computed by a linear addition of the multiples of the maximum transverse displacements of two such intersecting systems, i.e.

$$\delta_u = \alpha_x \delta_y + \beta_y \delta_x, \quad (3)$$

where α_x and β_y are dimensionless multipliers associated with the sequence and locations of the failing elements, i.e., the order of propagation of plasticity within the members of the structure. In fact, it will be shown that one of the two quantities α_x and β_y is always ± 1 and the other a fraction of the number of bays such as m , $m/2$ or $(m-1)/2$, depending on the boundary conditions and other physical properties of the system. The pair of Eqs. (2) and (3) together reduce the task of otherwise cumbersome plastic analysis to simple but meaningful additions of numbers. As practical applications of the LSC, consider the results of Tables 2 and 3.

As a general guideline, the interested reader may refer to (Grigorian 2013c) where a complete summary of the admissible failure patterns of grid type systems with all combinations of basic boundary support conditions has



been provided. The failure patterns of Table 1 of this reference are consistent with the individual beam or truss mechanisms and the corresponding boundary support conditions displayed along the top row and the right hand side margins of the same table. The locations of the first and last plastic hinges as well as the positions of the first and last failing elements, beams or trusses are also indicated for all combinations of boundary conditions.

The constitutive equilibrium equations

The key to the determination of plastic displacements of space trusses at incipient collapse is in the exact determination of the distributions of the internal forces and the location of the first and last sets of plastic deformations prior to failure. The challenge therefore, lays in the establishment of the corresponding constitutive equations of equilibrium for the type and class of structure under study.

A depiction of the horizontal projections of the axial forces acting on two representative adjacent joints of the subject space frame is presented in Fig. 4, where eight groups of interdependent force fields $C_{x,y}$, $\bar{C}_{x,y}$, $T_{x,y}$ and $\bar{T}_{x,y}$, $F_{x,y}$, $R_{x,y}$, $Q_{x,y}$ and $W_{x,y}$ have been identified. The cyclic formation of regular space frames allows their

constitutive difference equations of equilibrium (Renton 1964; Wah 1969) to be expressed in matrix form, whence the conditions of static equilibrium in the three mutually perpendicular directions, X , Y and Z at two neighboring joints x , y and $(x + \frac{1}{2})$, $(y + \frac{1}{2})$ give:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & E_y & E_x E_y & E_x & 0 & 0 \\ 1 & E_y & -E_x E_y & -E_x & 0 & 0 \\ 1 & -E_y & -E_x E_y & E_x & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{x,y} \\ R_{x,y} \\ F_{x,y} \\ W_{x,y} \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} -P_{x,y}/\theta \\ \nabla_x C_{x,y}/\psi \\ \nabla_y C_{x,y}/\psi \\ 0 \\ \Delta_x T_{x,y}/\psi \\ \Delta_x \bar{T}_{x,y}/\phi \end{bmatrix} \quad (4)$$

where the finite difference operator E_x^\pm performs the operation $E_x^\pm F(x) = F(x \pm 1)$ on any function of the variable x . Here, E and E^{-1} are the finite difference shift operators and imply $Ex = x + 1$ and $E^{-1}x = x - 1$, respectively. $\Delta = (E + 1)$ and $\nabla = (1 - E^{-1})$ are the forward and backward difference operators, respectively. The difference equilibrium matrix (4) can be reduced to the twin governing difference equations of equilibrium as:

Table 2 Maximum normal nodal displacements of the first and last failing trusses

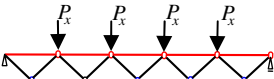
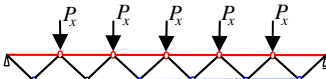
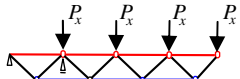
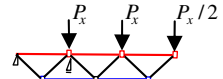
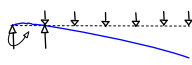
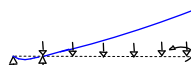
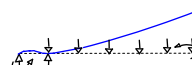
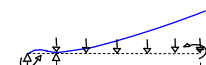
| | Last failing truss $m = \text{odd}$ | Last failing truss $m = \text{even}$ | First failing truss $m = \text{odd}$ | First failing truss $m = \text{even}$ |
|----------------------------------|---|---|--|--|
| $\frac{\delta E}{Pa^3}$ |  |  |  |  |
| $\frac{16\delta_c E}{Pa^3}$ | $\frac{1}{A_C h^2} \left[\frac{(m^2-1)(5m^2-9)}{24} \right]$ | $\frac{1}{A_C h^2} \left[\frac{m^2(5m^2-8)}{24} \right]$ | $\frac{(n-3)}{A_C} \left[\frac{3n^3+n^2-15n+11}{24} \right]$ | $\frac{(n-2)(n-3)}{A_C h^2} \left[\frac{n^2+2n-4}{4} \right]$ |
| $\frac{16\delta_r E}{Pa^3}$ | $\frac{1}{A_T h^2} \left[\frac{(m^2-1)(5m^2+3)}{24} \right]$ | $\frac{1}{A_T h^2} \left[\frac{m^2(5m^2+4)}{24} \right]$ | $\frac{(n-1)(n-3)}{A_T} \left[\frac{3n^2+4n+1}{24} \right]$ | $\frac{4}{A_T h^2} \left[\left(\frac{n}{2} \right)^2 \left(\frac{n}{2} - 1 \right)^2 \right]$ |
| $\frac{\theta^3 \delta_s E}{Ph}$ | $\frac{1}{A_S} \left[\frac{m^2-1}{4} \right]$ | $\frac{1}{A_S} \left[\frac{m^2}{4} \right]$ | $\frac{1}{A_S} \left[\frac{3n^2-8n+5}{4} \right]$ | $\frac{1}{A_S} \left[\frac{(5n^2-14n+8)}{4} \right]$ |

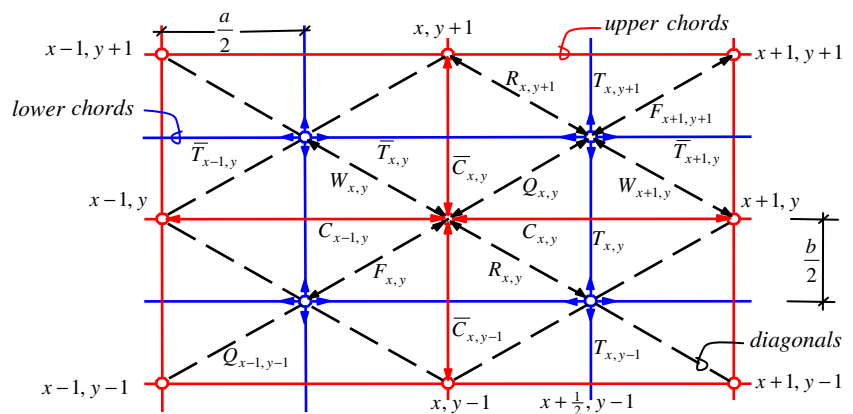
Table 3 Truss displacements and values of α_x and β_y for different boundary conditions

| Boundary conditions | Fix-free | Pin-pin | Fix-fix | Fix-pin |
|------------------------------------|---|---|--|---|
| Upper chord displacement |  |  |  |  |
| Outrigger length $m = \text{even}$ | $m \geq 1$ | $-m/2 \geq 2$ | $-(m-1)/2 \geq 2$ | $-(m-i) \geq 2$ |
| Outrigger length $m = \text{odd}$ | $m \geq 1$ | $-(m-1)/2 \geq 3$ | $-(m-1)/2 \geq 3$ | $-(m-i) \geq 3$ |

Where $z = (m - i - 1)$ and i is the nearest uprounded whole number to $[-(2m+1) + \sqrt{8m^2+1}]/2$



Fig. 4 Representation of axial forces on the horizontal projection of a regular basic module



$$\frac{1}{\psi} \Delta_x \nabla_x C_{x,y} + \frac{1}{\phi} \Delta_y \nabla_y \bar{C}_{x,y} = (\Delta_x + \Delta_y)(F_{x,y} - Q_{x,y}) + (\Delta_x - \Delta_y)(W_{x,y} - R_{x,y}) \quad (5)$$

$$\frac{1}{\psi} \Delta_x \nabla_x T_{x,y} + \frac{1}{\phi} \Delta_y \nabla_y \bar{T}_{x,y} = (\Delta_x + \Delta_y)(E_x E_y F_{x,y} - Q_{x,y}) + (\nabla_x - \nabla_y)(E_x W_{x,y} - E_y R_{x,y}). \quad (6)$$

The admissible force field solutions

While the set of Eqs. (4)–(6) is equally valid for all combinations of boundary support conditions, only the complete analysis of a generalized rectangular space truss, simply supported on all four sides, is presented for the purposes of the current discourse. The complete closed form solutions of all eight force fields defined for the subject truss system can be presented as follows;

$$C_{x,y} = \frac{4C}{m^2 + \partial_m + \gamma_m} \left[(m-1)x - x^2 + \frac{1}{2}(m-1) \right] \quad (7)$$

$$\bar{C}_{x,y} = \frac{4\bar{C}}{n^2 + \partial_n + \gamma_n} \left[(n-1)y - y^2 + \frac{1}{2}(n-1) \right] \quad (8)$$

$$T_{x,y} = \frac{4T}{m^2 + \partial_m} [mx - x^2] \quad (9)$$

$$\bar{T}_{x,y} = \frac{4\bar{T}}{n^2 + \partial_n} [ny - y^2]. \quad (10)$$

The basic form of the first line of the equilibrium Eq. (4), the uncoupled nature of group of Eqs. (7)–(10), and the fact that the system is composed of twistless square base pyramids suggest that the collapse load intensity $P_{x,y}$ may be assumed to be composed of two independent components P_x and P_y in the X and Y directions, respectively, i.e., $P_{x,y} = P_{x,C} + P_{y,C}$ and/or $P_{x,y} = P_{x,T} + P_{y,T}$ (11)

where subscripts C and T refer to compression and tension chord members, respectively. Solution (11) is in complete conformance with the principles of the LSC. It follows,

therefore, that the corresponding axial forces in the inclined web members may be expressed as;

$$Q_{x,y} = \frac{-1}{4\theta} [(m-1-2x)P_x + (n-1-2y)P_y] \quad m-1 \geq x \geq 0 \quad n-1 \geq y \geq 0 \quad (12)$$

$$F_{x,y} = \frac{1}{4\theta} [(m+1-2x)P_x + (n+1-2y)P_y] \quad m \geq x \geq 1 \quad n \geq y \geq 1 \quad (13)$$

$$R_{x,y} = \frac{-1}{4\theta} [(m-1-2x)P_x - (n+1-2y)P_y] \quad m-1 \geq x \geq 0 \quad n \geq y \geq 1 \quad (14)$$

$$W_{x,y} = \frac{1}{4\theta} [(m+1-2x)P_x - (n-1-2y)P_y] \quad m \geq x \geq 1 \quad n-1 \geq y \geq 0 \quad (15)$$

where the auxiliary terms;

$$\partial_m = [(-1)^m - 1]/2, \quad \partial_n = [(-1)^n - 1]/2, \quad \gamma_m = [(-1)^{m+1} - 1] \quad \text{and} \quad \gamma_n = [(-1)^{n+1} - 1] \quad (16)$$

have been introduced to generalize the proposed solutions for all combinations of even and odd number of bays in the two directions. As expected, the back substitution of Eqs. (7)–(15) results in the verification of the proposed solutions. Next, substituting for the relevant force functions in Eqs. (3) and (4) and performing the appropriate difference operations give after some manipulation;

$$P_{x,y,C} = \left[\frac{1}{a(m^2 + \partial_m + \gamma_m)} + \frac{1}{b(n^2 + \partial_n + \gamma_n)} \right] 8hC = P_{x,C} + P_{y,C} \quad (17)$$

$$P_{x,y,T} = \left[\frac{1}{a(m^2 + \partial_m)} + \frac{1}{b(n^2 + \partial_n)} \right] 8hT = P_{x,T} + P_{y,T} \quad (18)$$

as the carrying capacities of the subject DLG in terms of the compressive and tensile strengths of the upper and



lower chords, respectively. Once again the results of Eqs. (17) and (18) can be seen to be in conformity with the predictions of the LSC. With P_x and P_y known the corresponding forces of the diagonals can be computed using Eqs. (12)–(15). The usefulness of the proposed unique and exact solutions is not limited to assessing the true collapse load and the corresponding distributions of the internal forces but, in addition, they may be used to compute the reactions along the supports, i.e.,

$$V_{0,y} = (W_{1,y} + F_{1,y})\theta = (2m + \gamma_m)\frac{P_x}{4} \quad (19)$$

$$V_{m,y} = -(W_{m,y} + F_{m,y})\theta = (2m + \gamma_m)\frac{P_x}{4} \quad (20)$$

$$V_{x,0} = (F_{x,1} + R_{x,1})\theta = (2n + \gamma_n)\frac{P_y}{4} \quad (21)$$

$$V_{x,n} = -(F_{x,n} + R_{x,n})\theta = (2n + \gamma_n)\frac{P_y}{4}. \quad (22)$$

It is instructive to note that Eqs. (7) through (22) not only satisfy the prescribed failure criteria and static equilibrium, but also remain compatible with the stipulated boundary support conditions. The validity of the proposed solutions may be verified by summing the reactions along the supports, i.e.,

$$(m-1)(n-1)P_{x,y}. \quad (23)$$

The determination of maximum displacements associated with $P_{x,y}$ at the onset of failure is a function of the location of the first and last sets of yielding elements forming within the lower chords of the truss and is discussed in the following section.

Displacement analysis at incipient collapse

Because of the twistlessness of the basic module, i.e., a rectangular base pentahedron, the displacement analysis may be reduced to statically determinate conditions following the assumptions of the LSC, symbolized by Eqs. (2) and (3), provided that the correct sequences of formations of the yielded elements can be identified without resorting to cumbersome computations.

The rationale supporting the methodic positioning of the first and last sets of failing elements and the identification of the corresponding outrigger trusses, for the class of twistless grid systems are based on the consistency of research findings described in Appendix 2 as well as computer generated data. The formulation of proposed space truss deformation theorem has been facilitated immensely by the introduction of the equivalent linearized two-dimensional trusses. A linearized truss is one in which the upper chord consists of a string of pin jointed prismatic

members of cross-sectional areas A_C , the lower chord is made out of two parallel strings of pin jointed prismatic elements of cross-sectional areas $A_T/2$ each, and two sets of identical diagonals of cross-sectional areas $A_S/2$ that connect the two lower chords to the common upper chord.

The maximum normal nodal displacements of the first and last failing, equivalent, linearized, simply supported trusses with both odd and even numbers of bays have been summarized in Table 2, where δ_C , δ_T and δ_S represent the components of the maximum vertical displacement of the nodes of the top chord due to upper, lower and diagonal member stresses, respectively. Therefore, the maximum displacements of the two representative trusses in the X and Y directions can be estimated as:

$$\delta_x = \delta_{Cx} + \delta_{Tx} + \delta_{Sx} \quad \text{and} \quad \delta_y = \delta_{Cy} + \delta_{Ty} + \delta_{Sy}. \quad (24)$$

Obviously, equations similar to those provided in Table 2 can also be derived for all other boundary support conditions.

The theorem

An in-depth study of the failure patterns and characteristics of several groups of grillages (Grigorian 2013c), as well as the deformations of the first failing or outrigger trusses of Table 3 suggest that the estimation of the maximum displacements of the subject group of trusses at incipient collapse is governed by a single, general rule of application involving the maximum plastic displacements of two representative linearized, two dimensional intersecting trusses: in other words,

The maximum out of plane displacement of regular twistless space trusses, supported along the sides of a parallelogram and subjected to a uniform distribution of normal nodal forces at incipient collapse is given by the sum of the maximum displacements of the last remaining linearized truss of the more flexible set and α_x or β_y times the maximum displacement of first failing linearized truss of the other set, i.e.

$$\delta_u = \alpha_x \delta_y + \delta_x \quad \text{or} \quad \delta_u = \beta_y \delta_x + \delta_y \quad (25)$$

depending on the relative physical properties of the two sets of intersecting linearized trusses. Here, $\alpha_x \leq m$ or $\beta_y \leq n$ is the integer describing the location of the first failing member or the length of the outrigger truss, Fig. 3, where it has been assumed that the X direction linearized truss along $y = n/2$ is the first to fail and that the Y direction truss along $x = 1$ remains elastic until the onset of plastic failure.

Assuming tension failure, the deformed shapes of the upper chords of equivalent linearized trusses together with



the corresponding values of α_x and β_y and their limiting values for all practical combinations of boundary support conditions can be summarized as presented in Table 3.

It may be seen from Fig. 5 and Table 3 that except for free boundaries, α_x and β_y are always negative, implying that for non-free boundaries the curvature of the first failing truss is concave, i.e., the tip of the outrigger truss tends to curl upwards at collapse.

The first and last yielding members are the constituent parts of the first and last failing trusses, respectively. The first failing truss also contains the point of the maximum nodal displacement. Since the sequence of propagation of plasticity is related to the decreasing order of the stiffness of the two sets as well as the decreasing share of the interactive shear forces, then the first failing truss could be identified as the most heavily loaded truss of the stiffer of the two sets of intersecting trusses. The basic applications of Tables 2 and 3 are illustrated by the forthcoming demonstrative examples.

Displacements of a rectangular DLG

The novelty of the proposed theorem is that once the user becomes familiar with the applications of Tables 2, and 3, the manual solution of a DLG, such as the example problem, becomes easier and faster than feeding the corresponding data to a computer.

As an example of the applications of the proposed theorem consider the determination of the maximum plastic joint displacements of a generalized, $m \times n$, twistless, orthotropic DLG of regular formation subjected to monotonously increasing uniform distribution of normal nodal forces, with sets of simply supported trusses intersecting each other at constant angles. Assume tension type failure and even number of bays in both directions. Furthermore, assume that the X direction trusses are stiffer than those running in the Y direction. Because of double symmetry, the first failing truss may be identified as that laying along $y = n/2$ and the last failing trusses as those running along $x = 1$ and $x = m - 1$, respectively.

Solution From the LSC, Eq. (18), the ultimate uniform carrying capacity can be computed as:

$$P_{x,y,T} = \left[\frac{1}{am^2} + \frac{\mu}{bn^2} \right] 8hT = P_{x,T} + P_{y,T} \quad (26)$$

with P_x and P_y known, the 3rd column of Table 2 gives for the last failing truss:

$$\delta_y = \frac{P_y b^3}{16h^2 \bar{A}_C E} \left[\frac{n^2(5n^2 - 8)}{24} \right] + \frac{P_y b^3}{16h^2 \bar{A}_T E} \left[\frac{n^2(5n^2 + 4)}{24} \right] + \frac{4P_y h}{16\theta^3 \bar{A}_s E} n^2. \quad (27)$$

Similarly, the 5th column of Table 2 gives:

$$\delta_x = \frac{P_x a^3}{16h^2 \bar{A}_C E} (m^2 - 5m + 6) \left[\frac{m^2 + 2m - 4}{4} \right] + \frac{P_x a^3}{4h^2 \bar{A}_T E} \left[\left(\frac{m}{2} \right)^2 \left(\frac{m}{2} - 1 \right)^2 \right] + \frac{P_x h}{\theta^3 \bar{A}_s E} \left[\frac{5m^2 - 14m + 8}{4} \right] \quad (28)$$

and from Eq. (24) and column 3 of Table 3 $\alpha_x = m/2$, thus:

$$\delta_u = \frac{m}{2} \delta_y - \delta_x. \quad (29)$$

Equations (26) through (29) contain the complete solution for the problem under consideration. A numerical verification of the validity of these solutions is provided below.

Example 2-numerical verification

Use the proposed theorem to compute the maximum normal nodal displacements of a square, simply supported, isotropic, DLG of regular formation at incipient collapse and compare the results against those of a computer generated solution. The computer analysis may be simplified by providing continuous buckling restraints for all the top chord members. Let $P_{x,y} = 88.96$ kN (20 kips),

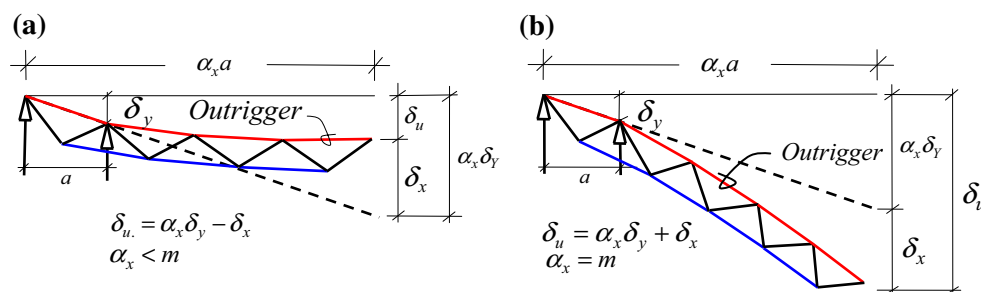


Fig. 5 Displacements of outriggers at collapse, **a** outrigger trusses with no free edges, **b** cantilevered outrigger



$m = n = 6$, $A_C = \bar{A}_C = A_T = \bar{A}_T = 21713 \text{ mm}^3$ (3.365 in^2), $A_S = \bar{A}_S = 21717 \text{ mm}^2$ (3.365 in^2), $\theta = 1/\sqrt{3}$, $a = b = 1,219.2 \text{ mm}$ (4 ft), and $E = 199,949.2 \text{ MPa}$ ($29,000 \text{ ksi}$).

Solution From Eq. (26) $P_x = P_y = P = 44.48 \text{ kN}$ (10 kips).

From Eq. (27); $\delta_y = \frac{129 \text{ Pa}}{2A_C E} + \frac{138 \text{ Pa}}{2A_T E} + \frac{27\sqrt{3} \text{ Pa}}{2A_S E} = 19.61 \text{ mm}$ (0.772 in.) and from Eq. (28); $\delta_x = \frac{33 \text{ Pa}}{A_C E} + \frac{36 \text{ Pa}}{A_T E} + \frac{9 \text{ Pa}\sqrt{3}}{A_S E} = 10.57 \text{ mm}$ (0.416 in.). From the displacement theorem, Eq. (29), $\delta_u = \frac{m}{2} \delta_y - \delta_x = 3 \times 19.61 - 10.57 = 48.26 \text{ mm}$ (1.90 in.), is an exact match with the computer generated result. It follows, therefore, that to control the maximum displacements of the structure at incipient collapse, any one or combination of the variables in Eqs. (27) and (28) can be modified, provided that the prescribed design criteria are not violated.

Concluding remarks

An analytic procedure has been presented that attempts to facilitate and revive interest in full elasto-plastic design of double-layer grids and similar structures under gravity loading conditions. PC is a direct design method that uses pre-selected target displacement and failure mechanism as key control objectives that relate the response of the structure to expected modes of behavior.

A new theorem has been presented which indicates that the exact displacement profiles at incipient collapse are associated with the unique limit state mechanisms, predicted by the LSC; in other words, the normal nodal displacements reach a maximum, when the collapse load intensity of the grid work becomes the sum of the collapse load intensities of the typical trusses of each direction.

It has been argued that the largest and smallest plastic extensions correspond to the first and last sets of yielding elements that forms within the first and last failing trusses, respectively. Therefore, the maximum nodal displacement at incipient collapse is governed by the magnitudes of the maximum displacements of the first and last collapsing trusses. Therefore, it would be reasonable to assume that the sequence of formation of the yielding members would be in the same order as the decreasing rotations of the failing trusses. This corroborates the notion that in flat, regular, three-dimensional structures of uniform sections, such as interconnected trusses, flexural grillages and reinforced concrete slabs the first set of yielding members, plastic hinges or initial lengths of yield lines form within segments or elements that contain the point of maximum normal elastic displacements. The last set of yielding elements, plastic hinges or lengths of yield lines form within the last standing members or segments of the structure prior to collapse.

It has been shown that multi-member DLG may benefit from a full elasto-plastic analysis where the limit load-carrying capacity and ultimate behavior of the structure can be accurately determined. To this end, the paper attempts to introduce PC as a viable methodology which may pave the way to designing more realistic and safer systems in the future. It has been shown that a simple way to control the strength and stiffness of such space frames subjected to any distribution of nodal forces is to control the strength and stiffness of groups of members under similar force fields. The main advantage of the proposed concept over traditional methods of design is its ability to provide more economic solutions while maintaining displacement control throughout the loading history of the structure. The proposed procedures, as they stand, are particularly useful for both preliminary as well as final design of ductile DLG. The proposed solutions lend themselves well to manual as well as spreadsheet computations. Most importantly, they help gain insight into the structural behavior of DLG at incipient plastic failure.

While the scope of this contribution is limited to a certain class of DLG; under gravity loading, the proposed methodology can be extended to other types of interconnected space structural systems and dynamic loading scenarios. However, in case of seismic excitations, additional attention should be paid to the ductility demands and capacities of the constituent members of such space structures. Since for large span roofs the effects of vertical components of earthquakes could be comparable with design level gravity forces.

Abbreviations All abbreviations and notation have been defined as they first appear in the text.

Conflict of interest The author declares that he has no competing interests.

Authors' contributions The author has been involved in space structures research for well over 40 years. His latest contributions include the introduction of performance control as an alternative design philosophy for ductile structures.

Authors' information The author is an international consultant, investigator and lecturer. He has more than 45 years of experience in structural design, education, code and standard development, forensic studies, research and project management. He holds B.Sc. and M.Sc. degrees in Structural Engineering from the University of Manchester (UMIST) and a D.Phil. degree in engineering from the University of Oxford. He is a California licensed Civil and Structural Engineer and is an active member of the Structural Engineers association of Southern California (SEAOC). He is a founding member and the first chairman of the Faculty of Structural Engineering (Sazeh) of Sharif University in Iran. Currently he is the Chief Structural Engineer of MGA Structural Engineering Consultants Inc. based on Glendale, CA, USA. In addition to his technical activities, he maintains an active social role that focuses on Martial Arts and Human Behavior. His latest books, *In Search of the Mind*, and *Karate-Do a Way of Life*, deal mainly with behavioral aspects of human life and social



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Appendix 1: Compression design formulae

For compression elements, member characteristics can be controlled against any approved code criteria, the Melbourne regression formula; $\frac{P_C}{P_Y} = 0.9 \frac{P_R}{P_Y} + 0.162$ or Merchants semi-empirical equation; $\frac{P_C}{P_Y} = 0.82 \frac{P_R}{P_Y} + 0.172$, where in both cases P_R is the Rankin load given by $\frac{1}{P_R} = \frac{1}{P_Y} + \frac{1}{P_E}$ in which P_Y and P_E are the squash or yield and elastic critical (Euler) loads, respectively. P_C is the chord or diagonal limit state compression force.

Appendix 2: Research findings

The rationale supporting the methodic positioning of the first and last sets of yielding members and the identification of the corresponding first failing trusses (the outrigger elements) for the load-specific class of subject space structures has been based on the consistency of the following findings and observations (Grigorian 2013c), that;

- The node containing the maximum normal displacements and the location of the last set of yielding member lay along the same outrigger.
- The first and last failing trusses cross each other at the angle of the parallelogram. The outrigger containing the maximum nodal displacement undergoes the largest plastic yielding at collapse.
- The first failing truss is identified as the most heavily loaded one of the stiffer of the two sets of intersecting trusses. The most heavily loaded trusses are those that span planes of zero interactive shear forces, e.g., a central truss that lays in a plane of symmetry, or act as end trusses that span a free edge.
- The first failing truss also contains the first set of plastic hinges and the point of maximum normal nodal displacement.
- The constituent trusses of a regular space truss fail in order of decreasing share of interactive nodal shear forces.
- The last failing truss contains the position of the last yielding member prior to system collapse, and can be identified as the least heavily loaded member of the either set.
- The locations of formation of the first and last yielding members are independent of the total number of active, inactive and simultaneously forming yielding elements within the structure.
- The displaced volume at incipient collapse is geometrically similar to the displaced volume caused by the corresponding failure mechanisms, and that the locations of maximum normal displacements are the same as the low points of the corresponding failure patterns.
- For grillages with one free edge, the first and last failing trusses run parallel to each other, and the outriggers containing the point of maximum displacement develop hogging curvatures.

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